

2.1 Prove the following formulas for the cross product: For any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$, show that:

- (i) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a}$ (Grassmann's 1st identity),
- (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{c}$ (Grassmann 2nd identity).
- (iii) $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle \langle \mathbf{b}, \mathbf{c} \rangle$ (Lagrange's identity).
- (iv) $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{d} \rangle$.

Hint. You might want to simplify your calculations by choosing an orthonormal basis well suited to the problem.

2.2 Is the cross product on \mathbb{R}^3 associative? If yes, prove it, else find a counterexample.

2.3 Show that for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$, we have

- (i) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ (Jacobi's identity)
- (ii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}]$,
where $[\cdot, \cdot, \cdot]$ denotes the *mixed product*.

Remark. The Jacobi identity for the cross product, together with the fact that $\mathbf{a} \times \mathbf{b}$ is bilinear and antisymmetric in \mathbf{a}, \mathbf{b} , implies that (\mathbb{R}^3, \times) has the structure of a *Lie algebra*.

2.4 For any *unit* vector $\mathbf{e} \in \mathbb{E}^3$ and any $\theta \in [-\pi, \pi]$, we will define by $R_\theta^{(\mathbf{e})} : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ the linear map which is the isometric rotation around the axis in the direction of \mathbf{e} by the angle θ , with the orientation convention that this is a “counter clockwise” rotation (i.e. if one points their right hand thumb in the direction of \mathbf{e} , the rotation for $\theta > 0$ is in the direction of the rest of the fingers). Show that, for any $\mathbf{v} \in \mathbb{E}^3$, we have

$$\left. \frac{d}{d\theta} R_\theta^{(\mathbf{e})}(\mathbf{v}) \right|_{\theta=0} = \mathbf{e} \times \mathbf{v}.$$

Remark. The above relation can be phrased as the statement that the operator $\mathbf{e} \times : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ is the infinitesimal generator of the group of rotations around e . In the language of Lie groups, this will mean that the (\mathbb{R}^3, \times) is the Lie algebra associated to the Lie group $SO(3)$.

2.5 In this exercise, we will prove a few statements about similarity transformations of a Euclidean vector space (recall the definition from first week's lecture).

- (a) Prove that the similarity transformations of a Euclidean vector space \mathbb{E}^n form a group.

- (b) Prove that the isometries form a normal subgroup of the group of similarity transformations.
- (c) Prove that the following properties are equivalent for an invertible linear map $f : \mathbb{E}^n \rightarrow \mathbb{E}^n$:
 1. f is a similarity transformation.
 2. f preserves angles, i.e. if $\mathbf{a}, \mathbf{b} \in \mathbb{E}^n$ are non-zero, then the angle between $f(\mathbf{a})$ and $f(\mathbf{b})$ is equal to the angle between \mathbf{a} and \mathbf{b} .
 3. f preserves orthogonality, i.e. if $\mathbf{a} \perp \mathbf{b}$ then $f(\mathbf{a}) \perp f(\mathbf{b})$.
- (d) We can identify \mathbb{C} with the oriented Euclidean plane \mathbb{R}^2 . Show that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a *linear* similarity transformation that preserves the orientation if and only if f is multiplication by a non-zero complex number (i.e. $f(z) = az$ with $a \in \mathbb{C}^*$). Similarly, if it changes the orientation, show that f is of the form $f(z) = a \cdot \bar{z}$. In both cases, show that f is a linear similarity transformation of scale $|a|$.

2.6 What are the conditions that a function $f : (0, 1) \rightarrow \mathbb{R}$ must satisfy so that its graph represents a biregular curve?

2.7 Recall that the length of a curve arc $\alpha : [a, b] \rightarrow \mathbb{R}^n$ is given by

$$\ell(\alpha) = \int_a^b V_\alpha(s) ds$$

where $V_\alpha(s) = \|\dot{\alpha}(s)\|$ is the speed of α .

Calculate the length of the following curves:

- (i) $\alpha(u) = (\cos(u), \sin(u), u)$ with $-\pi \leq u \leq \pi$
(the curve α is a right circular helix).
- (ii) $\beta(u) = (e^u, e^{-u}, \sqrt{2}u)$ with $0 \leq u \leq t$.
- (iii) $\gamma(u) = (u \cos(u), u \sin(u))$ with $0 \leq u \leq 4\pi$
(the curve γ is an Archimedean spiral).

2.8 The cycloid is the curve traced out by a point on the edge of a wheel that rolls, without slipping, in a straight line.

- (a) Draw a cycloid. Is it a smooth curve? Write down a parametrization of the cycloid.
- (b) Calculate the length of one arch of the cycloid (assuming the wheel generating the cycloid has radius r).